Fluctuations and correlations in sandpiles and interfaces with boundary pinning

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Interfaces are studied in an inhomogeneous critical state where boundary pinning is compensated with a ramped force. Sandpiles driven off the self-organized critical point provide an example of this ensemble in the Edwards-Wilkinson (EW) model of kinetic roughening. A crossover from quenched to thermal noise violates spatial and temporal translational invariances. The bulk temporal correlation functions have the effective exponents $\beta_{1D}\sim0.88\pm0.03$ and $\beta_{2D}\sim0.52\pm0.05$, while at the boundaries $\beta_{b,1D/2D}\sim0.47\pm0.05$. The bulk β_{1D} is shown to be reproduced in a randomly kicked thermal EW model.

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I. INTRODUCTION

Criticality in nonequilibrium systems is manifested in power-law scaling forms for various correlation functions. A particularly interesting class of behaviors exists in the kinetic roughening of interfaces. Even with common everyday phenomena there are considerable practical interests: a droplet of liquid on a porous substrate, an advancing crack in a material or a combustion front eating into an untouched material provide natural examples of rough, self-affine interfaces [1–3].

One frequent feature of systems which have apparent power-law scalings or critical properties is inhomogeneity [4]. Here we consider interface models in an ensemble where translational invariance is violated in the steady state. The main idea is that the drive (insertion of energy) and dissipation are spatially separated. This is relevant to growth processes in bounded systems, for which such an invariance [5] (both spatial and temporal) is usually assumed. We focus on rough interfaces with diffusional relaxation, with a quenched (frozen in-time) noise environment. An example is the quenched Edwards-Wilkinson universality class (QEW) [6–9]. A time-dependent force balances the presence of boundary conditions which pin the interface. The local interface velocity fluctuates strongly and often even vanishes, depending on the location. The two ingredients result in a violation of the spatial translational invariance and simultaneously complicate the temporal behavior [10]. One reason for this is that the local noise and the average local velocity couple, since the former is *a priori* independent of time, directly.

We first define in the following section the nonequilibrium steady state, in such a ''fixed drive-rate ensemble.'' Due to the broken nature of symmetries it is an open question as to what kind of criticality is to be expected. To tackle this we do in Sec. III a simple numerical analysis of temporal two-point correlations with such inhomogeneous fluctuations and couple these numerical observations with scaling arguments based on simple equivalent systems. Finally, we summarize the paper and remind ourselves of possible applications. These arise in systems where the local activity (order parameter) can be interpreted as such a ''velocity'' and can thus be studied over the entire time domain (quantities measurable in laboratory and space plasmas could provide examples [11]). In particular, this means that one considers the ''history'' of the local dynamics, like for cellular automata that map into interface models as *sandpile models* [10,12,13]. These can exhibit ''self-organized criticality'' (SOC) in the probability distributions of avalanche properties.

II. INTERFACES AND SANDPILES

The dynamics of a SOC model consists of individual avalanches, separated by addition of grains. In an avalanche, grains are shifted around by simple rules, and new sites get activated once a local threshold is exceeded, until one runs out of active sites. Due to an infinitesimal drive rate, a separation of time scales occurs. At the SOC critical point, the drive by adding ''grains'' is compensated by losses at the edges of the sandpile.

At faster drive rates, the avalanches overlap, the SOC-style criticality is destroyed, but the combination of a (random) drive and boundary dissipation still persists. Such dynamics in the SOC model(s) often maps to the QEW interface model [10,14]. Earlier work on sandpile fluctuations (see Refs. [15–17]) has concentrated on instantaneous quantities such as the local force or velocity, in interface terms (or activity and grains, in sandpile language). Likewise, their power spectra have been often studied, under the assumption of both spatial and temporal translational invariances. An analogy of why the right framework is essential here is provided by a simple random walk: the fluctuations of the *noise* could be measured (i.e., the walkers' step size), but they do not reveal the true (Langevin) equation of motion for the system. The Brownian particle is coincidentally a zero-dimensional interface model.

The time-integrated *activity* $H(x,t)=\int^t\rho(x,\tau)d\tau$ [18] in such sandpiles is, in several cases, described by the discrete QEW equation,

$$\partial_t H=\theta(\nu\nabla^2 H+\eta(x,H)+F(x,t)), \tag{1}$$

where the step function θ forces the interface velocity to be either zero or unity [14]. The time-dependent force $F(x,t)$ integrates the local drive (''energy''), the grains added to x

up to t, and forms a columnar noise term that changes slowly while being constant on the avalanche time scale [14].

The term η accounts for the quenched noise or randomness in the sandpile rules. We study two cases: a random threshold cellular automaton used to study the QEW critical properties [19], where the threshold $\eta(x,H) \equiv n_c(x,H)$ is randomly picked from a distribution, after each advance/toppling at x. The second choice is the Manna sandpile model [20] where two grains of sand are redistributed at random to neighbors of the site x, if the local "force" or the number of grains $n_x > n_c \equiv 1$.

With periodic boundary conditions, the QEW Eq. (1) has a *depinning* transition at an F_c. There the interface velocity vanishes, and critical correlations of the QEW universality class ensue. The interface is rough, characterized by self-affine temporal and spatial correlation functions [7–9]. If $F \gg F_c$, these models show a crossover to the "thermal" EW [with the standard construction $\eta(x,H) \rightarrow \eta(x,vt+\delta H)$] limit. The finite velocity washes out the quenched correlations in the noise.

Here, the off-criticality and the thermal limit are different. A boundary condition $H=0$ is imposed on the interface. In the steady state, the drive has to compensate for the increasing elastic energy, and thus the $F(x,t)$ term and the Laplacian match each other on the average. The choice $\langle F(x,t)\rangle = ft$ with f a fixed constant yields a constant average velocity which varies with x, $\langle v(x)\rangle \equiv \langle \partial_t H(x,t)\rangle$. The equation

$$\frac{\partial H(x,t)}{\partial t} = \nu \nabla^2 H(x,t) + ft + \delta f(x,t) + \eta(x,H(t)). \quad (2)$$

describes a depinning ensemble with a constant *drive rate f*, on average. Notice the difference with the usual "constant force," and in particular to the "constant velocity" scenario [21], which has sometimes been considered to describe SOC, in general. It is easy to see that the explicit time dependence in Eq. (2) can be absorbed by redefined variables $H \rightarrow H - \frac{1}{2}ft^2$ with the time dependence now being relegated to the boundary conditions. Under this change of variables, the mean-field height profile is a parabolic contour. We are interested in δH, the fluctuating part of $H(x,t)$. Equation (2) has a critical point $f_c(L)$ where the avalanches become distinguishable, with $f_c \rightarrow 0$ as L is increased. The most natural way to drive the system is to have spatially and temporally random increments $\Delta f(x,t)$ occurring on a time-scale $1/f$. This corresponds to the addition of discrete "grains" of force.

Just as $H(x,t)$, the velocity profile is inhomogeneous. The noise $\eta(x,v(x)+\delta H)$ develops temporal correlations that depend on x: translational invariance is *broken*. This is a fundamental property of the ensemble accounting for the fact that the boundary regions are closer to depinning than the bulk, as depicted in Fig. 1. In particular, both for the discrete QEW equation arising from the sandpile and for its continuum counterpart, $\lim_{x\rightarrow 0,L} v(x) = 0^+$. Due to the balance between elasticity and driving one has $\langle v\rangle \propto f/L^2$.

With a finite, small velocity $v(x)$, the effective noise-noise correlator reflects the avalanchelike dynamics. The interface often stays pinned $[v(x)=0]$ for a residence time τ

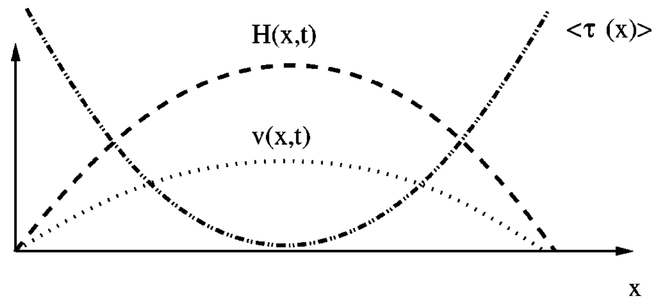

FIG. 1. A sketch of a one-dimensional (1D) interface, showing the velocity profile, the $\langle\tau\rangle$ (average "residence time"), and the height profile.

that varies. In the fixed drive rate ensemble τ has an x-dependent distribution $P(\tau,x)$, which defines the noise correlator $\langle \eta(x,H(x,t))\eta(x,H(x,t+\Delta t))\rangle$, since the contributions from the instances, when $H(x,t) \neq H(x,t+\Delta t)$, imply a δ-correlated noise field. An x-dependent P makes the analysis of the crossover from the depinning quite difficult, since an underlying statistical translational invariance is lacking [5,7–9].

In the simpler case of an interface with a point-correlated thermal noise $\eta(\vec{x},t)$ $[\langle \eta(\vec{x},t_1)\eta(\vec{y},t_2)\rangle = 2D\delta(\vec{x}-\vec{y})\delta(t_2 - t_1)$ where $2D$ is the noise strength] the temporal invariance is broken only for t small, when $v(x,t)$ is still time dependent, if the initial condition $H(x,0)=0$. For periodic or open boundary conditions, the Edwards-Wilkinson equation is statistically invariant if one simultaneously applies a drive producing a constant velocity $v(x,t) = \text{const}$, and the coordinate transformation $H(\vec{x},t) \rightarrow H - vt$. In such a steady state, with thermal, translation invariant noise, the fluctuating part of $H(x,t)$ separates and maps into a random walk with periodic boundary conditions. This is an example of return-to-zero properties of stochastic processes, analyzed recently by Baldassarri *et al.* [22].

The steady-state fluctuation amplitude $\langle(\delta H)^2\rangle_x$ is a function of x only and has an easy solution as the displacement of a walker returning to its starting point. The thermal exponent for the two-point time-correlation function, $\beta_{1D} = 1/4$, is to be compared with the constant drive-rate ensemble. The steady-state correlations or the dynamics of interface fluctuations are a measure of sandpile dynamics, which is different from correlations in avalanche properties [17], power-spectra of the activity [15,23], and, finally, the correlations in the activity or interface velocity itself [16].

III. INTERFACE FLUCTUATIONS

The interfaces are driven by depositing grains (adding to the force) at random locations, with a fixed rate so that new grains are added before the previous avalanche is over. The time is measured in discrete units, and a grain is deposited at every $1/f$ time step. In the steady state, the grains lost by toppling out of the system establish a balance between the external drive and the restoring elastic force. In one dimension (1D) we use the QEW cellular automaton, and in 2D, the Manna sandpile which, in this case, is close in the

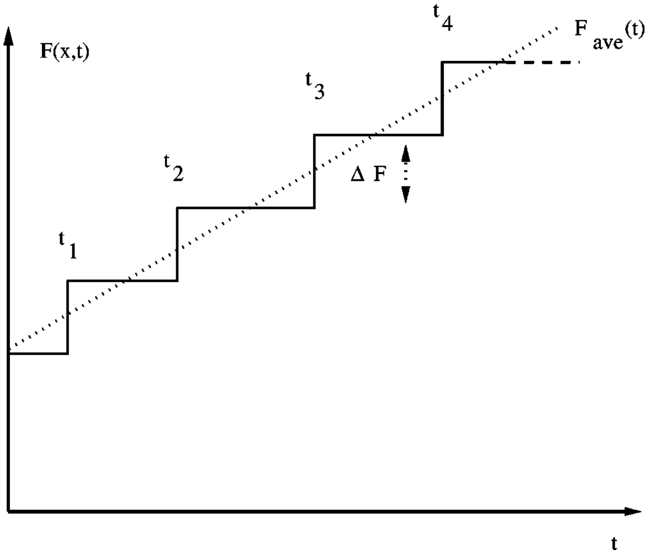

FIG. 2. Example of the driving force vs time.

depinning exponents to the 2D QEW scaling [24,25].

We study two properties: the temporal two-point correlation function $C(x,t)$ to investigate if the β exponent *can be defined* (via the usual scaling $C\sim t^{\beta}$) in spite of the lack of translational invariance of the noise η, and the local interface fluctuation amplitude $\langle(\delta H)^2\rangle$. The height field is decomposed into $H(x,t)=\langle H(x,t)\rangle+\delta H(x,t)$, such that $\langle H(x,t)\rangle=v(x)t+A(x)$ where the constant $A(x)$ accounts for the early time behavior, starting from $H(x,0)=0$ (see Fig. 3 for an example of δH). A statistical average is done over a window of time and a number of samples (Δt_{tot} up to 100 000, $N_{samples}$ 100 to 1000) to produce $C^2(x,t)=\langle|\delta H(x,t+t')-\delta H(x,t')|^2\rangle$. The local fluctuation amplitude scaling functions $\langle(\delta H(x))^2\rangle$ are also computed.

Close to the boundaries, $C(x,t)$ depends on the proximity to the depinning transition limit, i.e., $v(x)\sim 0^+$. In the bulk, loosely speaking, the noise becomes more thermal. In the central parts, $\langle v(x)\rangle\to 1^-$ takes place for f large enough, which makes $C(x,t)$ quasitrivial since the correlations in ΔH vanish. Note that for the QEW automaton (or the Manna model for that matter) $0\leqslant v\leqslant 1$ and if a site has always $n(x,t)>n_c(x,t)$ due to f being large [Eq. (2)], the extra force at site x is irrelevant.

The fluctuating part of the drive, $\delta f(x,t)$ in Eq. (2) is demonstrated in Fig. 2.

If in a fixed-drive rate ensemble the force $F(x,t)$ at a site x increases discontinuously, the "jerky" dynamics is

$$F(x,t)=\sum_{t_{i,x}}\Delta F_{t_{i,x}}(x)\,\theta(t-t_{i,x}),\qquad(3)$$

where $t_{i,x}$ are the (random) time instances at which the forces at sites x increase. Due to the piecewise constant nature of $F(x,t)$, a transformation to $F(x,t)=\langle F(t)\rangle+\delta F(x,t)$ produces an "impulsive" Newtonian force term $\delta F(x,t)$ that is piecewise *linear* in time, between the discontinuities at t_i. An analogy is thus given by the "kicked EW," k-EW, which ensues by applying a drive as in Eq. (3) to the thermal EW, in the same ensemble (see again Figs. 2 and 3).

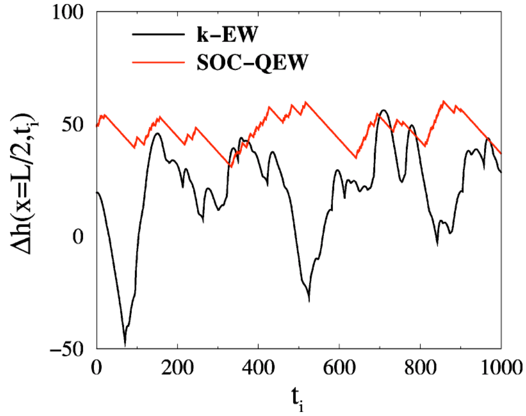

FIG. 3. Examples of the $\delta H(x,t)$ at the center of a 1D system for both a QEW sandpile and a kicked Edwards-Wilkinson model (see text).

The simplest analogy to this scenario is the deterministic relaxation of an elastic string, plucked at random locations. If the rate of vibration is large enough, it produces thermal noise. Now the modes of vibrations will exhibit the competing effect between the drive and the relaxation due to the restoring elastic force, similar to elastic interfaces in the presence of columnar (x dependent), quenched force fields [8]. EW model has the inherent symmetry that time-independent forcing maps into an initial profile, which relaxes deterministically, and thermal fluctuations are irrelevant. Resolving Eq. (2) into its mean and fluctuating parts, for a constant acceleration [$f(t)=$const], the time-correlation function can be shown to take the forms $\beta=(4-d)/4$, for all $t<t_c$, and $\beta=(2-d)/4$ (thermal EW value) for all $t>t_c$, where $t_c\sim 1/\nu f$. In low dimensions, where fluctuations govern the basic dynamics, the jerky drive adds to the SOC force, such that $F(t)=f_0+f_1 t$. This additional factor competes with relaxation, on the time scale $1/f$, in which $\delta f(x,t)$ is linear in time locally, and depends on the dimensionality. Some further analysis allows one to argue that in 1D, β should lie between 3/4 and 5/4.

Figure 4 shows the pertinent features of numerical experiments on $C(x,t)$. The correlation functions are presented for several locations and compared to the k-EW model. The time and space symmetries (both translational and reversal) are broken—compared to a pure EW model for which $\beta_{EW}=1/4$—and several *effective* exponents can be obtained. For $x\gg 1$, $\beta_{1D}\sim 0.88\pm 0.03$ depending on the exact fitting window. This compares nicely with the scaling of the k-EW case, and note that the value also agrees with the *critical* QEW β exponent, as determined numerically in 1D [7].

For the k-EW, the driving force is imposed by cycling the additions δF between different x, randomly, so that the k-EW drive fluctuations $\delta f=F(x,t)-ft$ stay bounded, whereas in the SOC case $\langle|\delta f|\rangle\sim t^{1/2}$ follows Poissonian statistics. Above the time scale t_c of the cyclic forcing, the k-EW has a crossover to usual EW scaling with $\beta\sim 1/4$.

In two dimensions, with the Manna model, $\beta_{2D}=0.52\pm 0.05$, i.e., closer to the lower limit of 1/2 (compare with

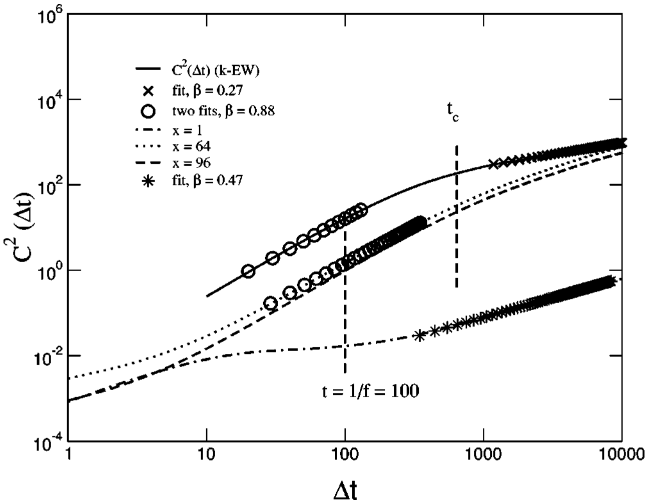

FIG. 4. Temporal two-point correlation functions for the 1D case for both the QEW/SOC automaton (for several locations x), $L=512$, and the kicked EW model.

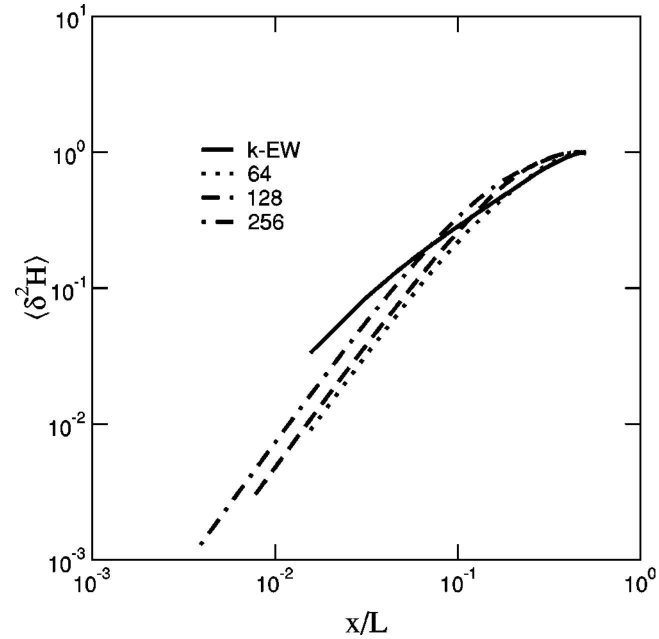

FIG. 5. Amplitude of the interface fluctuations (normalized at $x=L/2$) $\langle \delta H^2\rangle(x)$ in 1D for both the QEW and the kicked EW models. $f=1/100$ for the former for all the $L=64,\ldots,256$.

$\beta_{EW,2D}=0$). In central parts of such simple model systems, the restriction $v(x)\sim 1^-$ comes into play, and the fluctuations are constrained by the binary character of the velocity (i.e., in simple terms it is rare to see that $v=0$). For both 1D (depicted in Fig. 4) and 2D cases, the boundary behavior differs from the bulk. The fluctuations are driven by the dissipation of the energy inserted into the bulk and the ensuing elastic fluctuations dominate. This is described simply by $\partial \delta H(x,t)/\partial t \simeq \eta(x,t)$, with the result $C(x,t)\sim 2D(x)t$. These correlations at the pinned boundaries have a crossover to the temporal scaling in the bulk. The boundary system in this regime is technically equivalent to the *random-deposition* model of growth, which justifies the value $\beta_b \simeq 0.47\pm 0.05$ close to the boundaries.

The strong dependence of the fluctuations on x is illustrated further in Fig. 5, where the spatial profiles of $\langle(\delta H)^2\rangle(x)$ are depicted.

The fluctuation profiles show the same scaling behavior for all the system sizes as a function of x for the 1D QEW: $\langle(\delta H)^2\rangle(x)\sim x^{1.8}$. The k-EW exhibits the $\langle(\delta H)^2\rangle(x)\sim x$ dependence. This returning random-walk-like statistics is expected since the k-EW has a crossover to the EW phase for large t. It would be interesting to study such fluctuation profiles in $d>1$.

One can argue that the exponent arises due to simple translationally nonuniform noise. In the steady state, the local fluctuations are given by $\langle(\delta H)^2\rangle(x)\sim D(x)/\nu$, where $D(x)$ now denotes the effective thermal noise strength, at x. In this limit, the noise-noise correlation function $\langle \eta(x,H(x,t))\eta(x,H(x,t+\Delta t))\rangle \sim D(x)$ is proportional to the squared velocity fluctuations; since $v\sim x^2$ which implies that $\delta v\sim x$ and thus $D(x)\sim x^2$ which gives $\langle(\delta H)^2\rangle(x)\sim x^2$. This is reasonably close to the numerical value of 1.8. Please note that the model used tacitly assumes the restriction $0\leqslant v\leqslant 1$, implying a nonparabolic velocity profile close to the mid part of the interface. This simple scaling argument would then demand that in 1D, for any EW-like process where the velocity profile in the steady state is inhomogeneous, the fluctuation scaling function follows from similar considerations.

IV. SUMMARY

For an interface moving in a quenched random landscape, both the spatial and temporal translational invariances can be violated by boundary pinning, compensated by an external drive. A steady state arises automatically ("constant driverate"), with a linearly in-time increasing force, on the average. This also applies to sandpiles in the overlapping avalanche regime. We have concentrated on an intermittently increasing drive force since it seems the pertinent choice for most model systems and describes SOC models. If the driving were applied uniformly (as in, e.g., the Olami-Feder-Christensen earthquake model [26]), the inhomogeneous crossover to the thermal noise would still persist, leading to a violation of the translational invariance. This would also be the case if one considered a domain wall in a magnet, described by the QEW (driven uniformly by an applied field that increases and pinned at the ends).

The part of the correlation behavior that arises from the fluctuating drive is reproduced in a similarly driven thermal EW equation. The two-point temporal correlations develop a new effective scaling regime, resulting in different exponents for the boundary and the bulk. These scalings could be sought experimentally, e.g., directly in interface dynamics in random systems [27], or from the activity history of any system with intermittent, continuous dynamics (e.g., plasma physics including magnetospheric activity, reaction-diffusion systems [11,28]).

The ensemble presents theoretical problems in understanding the spatially varying crossover from the quenched noise, which is related to the concept of boundary critical phenomena [29] in interfaces. Similar occurrences of a breakdown of translational invariance might exist for other Langevin equations. An example is the one that describes an absorbing state phase transition, with history-dependent terms included [30], and in fact the 2D Manna model is expected to be in the same universality class. One can envision even more complicated scenarios, as the quenched Kardar-Parisi-Zhang equation. Imposing a steady state leads in that case to direct symmetry breaking in the SOC limit [31]. The boundary conditions are here more important in the thermal case, qualitatively speaking, than for the EW universality class. A study of the fixed drive rate case would seem interesting.

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